

THE ESCAPING SET OF A QUASIREGULAR MAPPING

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ABSTRACT. We show that if the maximum modulus of a quasiregular mapping $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ grows sufficiently rapidly then there exists a non-empty escaping set $I(f)$ consisting of points whose forward orbits under iteration of f tend to infinity. We also construct a quasiregular mapping for which the closure of $I(f)$ has a bounded component. This stands in contrast to the situation for entire functions in the complex plane, for which all components of the closure of $I(f)$ are unbounded, and where it is in fact conjectured that all components of $I(f)$ are unbounded.

MSC 2000: Primary 30C65, 30C62; secondary 37F10.

1. INTRODUCTION

In the study [1] of the dynamics of nonlinear entire functions $f : \mathbb{C} \rightarrow \mathbb{C}$ considerable recent attention has focussed on the escaping set

$$I(f) = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} f^n(z) = \infty\},$$

where $f^1 = f$, $f^{n+1} = f \circ f^n$ denote the iterates of f . Eremenko [5] proved that if f is transcendental then $I(f) \neq \emptyset$ and indeed that, in keeping with the nonlinear polynomial case [17], the boundary of $I(f)$ is the Julia set $J(f)$. The proof in [5] that $I(f)$ is non-empty is based on the Wiman-Valiron theory [6].

For transcendental entire functions f , Eremenko went on to prove in [5] that all components of the closure of $I(f)$ are unbounded, and to conjecture that the same is true of $I(f)$ itself. For entire functions with bounded postcritical set this conjecture was proved by Rempe [11], and for the general case it was shown by Rippon and Stallard [15] that $I(f)$ has at least one unbounded component.

In the meromorphic case the set $I(f)$ was first studied by Dominguez [4], who proved that again $I(f) \neq \emptyset$ and $\partial I(f) = J(f)$. For meromorphic f it is possible that all components of $I(f)$ are bounded [4], and the closure of $I(f)$ may have bounded components even if f has only one pole [4, p.229]. On the other hand $I(f)$ always has at least one unbounded component if the inverse function f^{-1} has a direct transcendental singularity over infinity: this was proved by Bergweiler, Rippon and Stallard [3] by developing an analogue of the Wiman-Valiron theory in the presence of a direct singularity.

The present paper is concerned with the escaping set for quasiregular mappings $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ [14], which represent a natural counterpart in higher real dimensions of analytic functions in the plane, and exhibit many analogous properties, a highlight among these being Rickman's Picard theorem for entire quasiregular maps

This research was supported by: the G.I.F, the German-Israeli Foundation for Scientific Research and Development, Grant G-809-234.6/2003, and the EU Research Training Network CODY (Bergweiler); EPSRC grant RA22AP(Fletcher and Langley); the ESF Research Networking Programme HCAA (Bergweiler and Langley); DFG grant ME 3198/1-1 (Meyer).

[12, 14]. For the precise definition and further properties of quasiregular mappings we refer the reader to Rickman's text [14]. Now the iterates of an entire quasiregular map are again quasiregular, and properties such as the existence of periodic points were investigated in [2, 16]. Further, there is increasing interest in the dynamics of quasiregular mappings on the compactification $\overline{\mathbb{R}^N}$ of \mathbb{R}^N , although attention has been restricted to mappings which are uniformly quasiregular in the sense that all iterates have a common bound on their dilatation: see [8, Section 21] and [7]. In the absence of this uniform quasiregularity there are evidently some difficulties in extending some concepts of complex dynamics to quasiregular mappings in general, but the escaping set $I(f)$ makes sense nevertheless, and we shall prove the following theorem.

Theorem 1.1. *Let $N \geq 2$ and $K > 1$. Then there exists $J > 1$, depending only on N and K , with the following property.*

Let $R > 0$ and let $f : D_R \rightarrow \mathbb{R}^N$ be a K -quasiregular mapping, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set

$$(1) \quad B_R = \{x \in \mathbb{R}^N : R \leq |x| < \infty\}.$$

Assume that f satisfies

$$(2) \quad \liminf_{r \rightarrow \infty} \frac{M(r, f)}{r} \geq J, \quad \text{where} \quad M(r, f) = \max\{|f(x)| : |x| = r\},$$

and define the escaping set by

$$(3) \quad I(f) = \{x \in \mathbb{R}^N : \lim_{n \rightarrow \infty} f^n(x) = \infty\}, \quad f^1 = f, \quad f^{n+1} = f \circ f^n.$$

Then $I(f)$ is non-empty. If, in addition, f is K -quasiregular on \mathbb{R}^N then $I(f)$ has an unbounded component.

The proof of Theorem 1.1 is based on the approach of Dominguez [4], as well as that of Rippon and Stallard [15]. A key role is played also by the analogue of Zalcman's lemma [18, 19] developed for quasiregular mappings by Miniowitz [9] (see §2). It seems worth observing that in Theorem 1.1 the hypothesis (2) cannot be replaced by

$$\liminf_{r \rightarrow \infty} \frac{M(r, f)}{r} > 1,$$

as the following example shows. Take cylindrical polar coordinates $r \cos \theta, r \sin \theta, x_3$ in \mathbb{R}^3 , let $\lambda > 0$ and let f be the mapping defined by

$$0 \rightarrow 0, \quad (re^{i\theta}, x_3) \rightarrow (re^{\lambda \cos \theta + i(\theta + \pi)}, x_3).$$

Then f^2 is given by

$$(re^{i\theta}, x_3) \rightarrow (re^{\lambda \cos \theta + \lambda \cos(\theta + \pi) + i(\theta + 2\pi)}, x_3)$$

and so is the identity, while since f is C^1 on $\mathbb{R}^3 \setminus \{0\}$ and satisfies $f(2x) = 2f(x)$ it is easy to see that f is quasiconformal on \mathbb{R}^3 . On the other hand if $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is quasiregular with an essential singularity at infinity, then $M(r, f)/r \rightarrow \infty$ as $r \rightarrow \infty$ (see, for example, [2, Lemma 3.4]) so that (2) holds with any $J > 1$.

Next, we show in §6 that there exists a quasiregular mapping f on \mathbb{R}^2 with an essential singularity at infinity, such that the closure of $I(f)$ has a bounded component. Thus while the result of [15] that $I(f)$ has at least one unbounded component extends to quasiregular mappings by Theorem 1.1, Eremenko's theorem [5] that all components of the closure of $I(f)$ are unbounded does not.

We remark finally that it is easy to show that if f is quasimeromorphic with infinitely many poles in \mathbb{R}^N then $I(f)$ is non-empty, and for completeness we outline how this is proved in §7, using the “jumping from pole to pole” method [3, 4].

2. THEOREMS OF RICKMAN AND MINIOWITZ

Let G be a domain in \mathbb{R}^N . A continuous mapping $f : G \rightarrow \mathbb{R}^N$ is called quasiregular [14] if f belongs to the Sobolev space $W_{N,loc}^1(G)$ and there exists $K \in [1, \infty)$ such that

$$|f'(x)|^N \leq K J_f \quad \text{a.e. in } G.$$

Moreover, f is called K -quasiregular if its inner and outer dilations do not exceed K : for the details and equivalent definitions we refer the reader to [14]. Rickman proved [12, 14] that given $N \geq 2$ and $K \geq 1$ there exists an integer $C(N, K)$ such that if f is K -quasiregular on \mathbb{R}^N and omits $C(N, K)$ distinct values $a_j \in \mathbb{R}^N$ then f is constant. Here $C(2, K) = 2$ because a quasiregular mapping in \mathbb{R}^2 may be written as the composition of a quasiconformal mapping with an entire function, but for $N \geq 3$ this integer $C(N, K)$ in general exceeds 2 [13, 14].

Miniowitz [9] established for quasiregular mappings the following direct analogue of Zalcman’s lemma [18, 19]. A family F of K -quasiregular mappings on the unit ball B^N of \mathbb{R}^N is not normal if and only if there exist

$$f_n \in F, \quad x_n \in B^N, \quad x_n \rightarrow \hat{x} \in B^N, \quad \rho_n \rightarrow 0+,$$

such that

$$f_n(x_n + \rho_n x) \rightarrow f(x)$$

locally uniformly in \mathbb{R}^N , where f is K -quasiregular and non-constant. Using this she established the following analogue of Montel’s theorem, in which $C(N, K)$ is the integer from Rickman’s theorem [12] and $\chi(x, y)$ denotes the spherical distance on \mathbb{R}^N .

Theorem 2.1 ([9]). *Let $N \geq 2, K > 1, \varepsilon > 0$ and let D be a domain in \mathbb{R}^N . Let F be a family of functions K -quasiregular on D with the following property. Each $f \in F$ omits $q = C(N, K)$ values $a_1(f), \dots, a_q(f)$ on D , which may depend on f but satisfy*

$$\chi(a_j(f), a_k(f)) \geq \varepsilon \quad \text{for } j \neq k.$$

Then F is normal on D .

Theorem 2.1 leads at once to the following standard lemma of Schottky type.

Lemma 2.1. *Let $N \geq 2$ and $K > 1$. Then there exists $Q > 2$ with the following property. Let f be K -quasiregular on the set $\{x \in \mathbb{R}^N : 1 < |x| < 4\}$ such that f omits $q = C(N, K)$ values y_1, \dots, y_q , with*

$$|y_j| = 4^{j-1}, \quad j = 1, \dots, q.$$

If $\min\{|f(x)| : |x| = 2\} \leq 2$ then $\max\{|f(x)| : |x| = 2\} \leq Q$.

3. TWO LEMMAS NEEDED FOR THEOREM 1.1

We need the following two facts, the first of which is from Newman’s book [10, Exercise, p.84]:

Lemma 3.1. *Let G be a continuum in $\overline{\mathbb{R}^N} = \mathbb{R}^N \cup \{\infty\}$ such that $\infty \in G$, and let H be a component of $\mathbb{R}^N \cap G$. Then H is unbounded.*

This leads on to the second fact we need:

Lemma 3.2. *Let E be a continuum in $\overline{\mathbb{R}^N}$ such that $\infty \in E$, and let $g : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a continuous open mapping. Then the preimage*

$$g^{-1}(E) = \{x \in \mathbb{R}^N : g(x) \in E\}$$

cannot have a bounded component.

For completeness we give a proof of Lemma 3.2 in §8.

4. AN ANALOGUE OF BOHR'S THEOREM

Let $f : D_R \rightarrow \mathbb{R}^N$ be K -quasiregular, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set B_R in (1), and assume that f satisfies (2) for some $J > 1$. For $0 \leq r < s \leq \infty$ set

$$A(r, s) = \{x \in \mathbb{R}^N : r < |x| < s\}.$$

Using (2) choose $s_0 > R$ such that

$$M(r, f) > M(R, f) \quad \text{for all } r \geq s_0.$$

Then $M(r, f)$ is strictly increasing on $[s_0, \infty)$ because if $s_0 \leq r_1 < r_2 < \infty$ and $M(r_2, f) \leq M(r_1, f)$ then $|f(x)|$ has a local maximum at some $\hat{x} \in A(R, r_2)$, which contradicts the openness of the mapping f . Following Dominguez [4] we establish a lemma analogous to Bohr's theorem.

Lemma 4.1. *Let $c = 1/2Q$, where Q is the constant of Lemma 2.1. Then for all sufficiently large ρ there exists $L \geq cM(\rho/2, f)$ such that*

$$S(0, L) = \{x \in \mathbb{R}^N : |x| = L\} \subseteq f(A(R, \rho)).$$

Proof. Using (2) let ρ be so large that

$$(4) \quad \rho > 4R \quad \text{and} \quad S = cM(\rho/2, f) > 2T = 4M(R, f),$$

and assume that the assertion of the lemma is false for ρ . Then for $j = 1, \dots, q$, where $q = C(N, K)$ is the integer from Rickman's Picard theorem [12] (see §2), there exists $a_j \in \mathbb{R}^N$ with

$$(5) \quad |a_j| = 4^{j-1}S \quad \text{and} \quad a_j \notin f(A(R, \rho)).$$

Furthermore, there exists $x_1 \in A(R, \rho/2)$ such that $|f(x_1)| = S$. To see this join a point x_0 on $S(0, \rho/2)$ such that $|f(x_0)| = M(\rho/2, f)$ to $S(0, R)$ by a radial segment and use (4) and the fact that $c < 1$. Let G be the component of the set

$$\{x \in \mathbb{R}^N : T < |f(x)| < 2S\}$$

which contains x_1 . Then $G \subseteq A(R, \infty)$ by (4). Suppose first that $G \subseteq A(R, \rho/2)$. Then the closure \overline{G} of G lies in $A(R, \rho)$, by (4) again. Choose a geodesic $\sigma \subseteq S(0, S)$ joining $f(x_1)$ to a_1 . Let

$$\mu = \inf\{|f(x) - a_1| : x \in \overline{G}, f(x) \in \sigma\}$$

and take $\zeta_n \in \overline{G}$ with $f(\zeta_n) \in \sigma$ and $|f(\zeta_n) - a_1| \rightarrow \mu$. Then we may assume that $\zeta_n \rightarrow \hat{\zeta} \in \overline{G}$, and we have $f(\hat{\zeta}) \in \sigma$ and so $\hat{\zeta} \in G$. But then the open mapping theorem forces $\mu = |f(\hat{\zeta}) - a_1| = 0$, which contradicts (5).

Thus $G \not\subseteq A(R, \rho/2)$ and this implies using (4) again that there must exist x_2 on $S(0, \rho/2)$ such that $|f(x_2)| \leq 2S$. By (4) and (5) the function $g(x) = f(x\rho/4)/S$ is K -quasiregular on $A(1, 4)$, and omits the q values $y_j = a_j/S$, which satisfy

$|y_j| = 4^{j-1}$. Since $|g(4x_2/\rho)| \leq 2$, Lemma 2.1 implies that $|g(x)| \leq Q$ for $|x| = 2$, which gives

$$M(\rho/2, f) \leq QS = QcM(\rho/2, f) = \frac{M(\rho/2, f)}{2},$$

a contradiction. \square

5. PROOF OF THEOREM 1.1

Again let $f : D_R \rightarrow \mathbb{R}^N$ be K -quasiregular, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set B_R in (1), but this time assume that f satisfies (2) for some large positive J . Retain the notation of §4. Following Dominguez' method [4] let $\rho_0 > R$ be so large that every $\rho \geq \rho_0$ satisfies the conclusion of Lemma 4.1 and further that, with the same constant c as in Lemma 4.1,

$$(6) \quad cM(\rho/2, f) > 4\rho > \rho > M(R, f) \quad \text{for all } \rho \geq \rho_0,$$

which is possible by (2) and the assumption that J is large. Fix $\rho \geq \rho_0$.

Lemma 5.1. *There exist bounded open sets G_0, G_1, \dots with the following properties.*

(i) *The set $\overline{\mathbb{R}^N} \setminus G_n$ has two components, namely*

$$\tilde{G}_n = \overline{B(0, R)} = \{x \in \mathbb{R}^N : |x| \leq R\}$$

and $G_n^ = A_n$, which satisfies $\infty \in A_n$.*

(ii) *We have*

$$(7) \quad \{x \in \mathbb{R}^N : R < |x| \leq 2^n \rho\} \subseteq G_n.$$

(iii) *The sets G_n , A_n and $\gamma_n = \partial A_n$ satisfy*

$$(8) \quad \gamma_{n+1} \subseteq f(\gamma_n) \quad \text{and} \quad f(G_n) \cap A_{n+1} = \emptyset.$$

Proof. The open sets G_n will be constructed inductively. We begin by setting $G_0 = A(R, \rho')$ for some $\rho' > \rho$, so that (7) obviously is satisfied for $n = 0$. It remains to show how to construct G_{n+1} given the existence of G_0, \dots, G_n for some $n \geq 0$. The fact that f maps open sets to open sets gives

$$(9) \quad \partial f(G_n) \subseteq f(\partial G_n) = f(S(0, R)) \cup f(\gamma_n),$$

using (i) and the definition $\gamma_n = \partial A_n$. By Lemma 4.1, (6) and (7) there exists

$$(10) \quad T_n \geq cM(2^{n-1}\rho, f) > 2^{n+2}\rho \quad \text{with} \quad S(0, T_n) \subseteq f(A(R, 2^n \rho)) \subseteq f(G_n).$$

Now $f(G_n)$ is a bounded open set, so let A_{n+1} be the component of $\overline{\mathbb{R}^N} \setminus f(G_n)$ which contains ∞ and set

$$(11) \quad \gamma_{n+1} = \partial A_{n+1}.$$

Then by (10) we have

$$(12) \quad \gamma_{n+1} \subseteq A_{n+1} \subseteq A(2^{n+2}\rho, \infty),$$

and (6), (9) and (11) imply the first assertion of (8). Let

$$G_{n+1} = \mathbb{R}^N \setminus (\overline{B(0, R)} \cup A_{n+1}).$$

Then (i) is satisfied with n replaced by $n+1$, and the second assertion of (8) follows from the definition of A_{n+1} . Finally (12) shows that (7) is satisfied with n replaced by $n+1$, and so the induction is complete. \square

Lemma 5.2. *Let $w \in \gamma_n$. Then there exists $z_n \in \gamma_0$ with $f^n(z_n) = w$ and*

$$(13) \quad f^m(z_n) \in \gamma_m \quad \text{for } m = 0, \dots, n.$$

Proof. This is easily proved using induction and (8). \square

Now take a sequence of points $z_n \in \gamma_0$ satisfying (13). We may assume that (z_n) converges to $\hat{z} \in \gamma_0$, and we have, by (13),

$$(14) \quad f^m(\hat{z}) = \lim_{n \rightarrow \infty} f^m(z_n) \in \gamma_m \quad \text{for each } m \geq 0.$$

Using (12) we get $\hat{z} \in I(f)$ and hence $I(f)$ is non-empty. This proves the first assertion of Theorem 1.1.

The second assertion will be established by modifying the method of Rippon and Stallard [15], so assume that f is K -quasiregular in \mathbb{R}^N and take \hat{z} satisfying (14). As before let $A_n = G_n^*$ be the component of $\overline{\mathbb{R}^N} \setminus G_n$ containing ∞ , and let L_n be the component of $f^{-n}(A_n)$ containing \hat{z} , which is well-defined since $f^n(\hat{z}) \in \gamma_n$ and $\gamma_n = \partial A_n$ by definition.

Lemma 5.3. *L_n is closed and unbounded.*

Proof. L_n is closed since A_n is closed, and L_n is unbounded by Lemma 3.2. \square

Lemma 5.4. *We have $L_{n+1} \subseteq L_n$ for $n = 0, 1, \dots$*

Proof. Suppose that $f^{n+1}(z') \in A_{n+1}$ but $f^n(z') \notin A_n$. Thus either $|f^n(z')| \leq R$ or $f^n(z') \in G_n$, from which we obtain $f^{n+1}(z') \notin A_{n+1}$, in the first case from (6) and (7) and in the second case from (8), and this is a contradiction. Hence if $z' \in L_{n+1}$ then z' lies in a component of $f^{-n-1}(A_{n+1})$ which contains \hat{z} , and this component in turn lies in a component of $f^{-n}(A_n)$. Hence we get $z' \in L_n$. \square

We may now write

$$K_n = L_n \cup \{\infty\}, \quad \{\hat{z}, \infty\} \subseteq K_{n+1} \subseteq K_n, \quad \{\hat{z}, \infty\} \subseteq K = \bigcap_{n=0}^{\infty} K_n.$$

Since K_n is compact and connected so is K [10, Theorem 5.3, p.81]. Let Γ be the component of $K \setminus \{\infty\}$ which contains \hat{z} . Then Γ is unbounded by Lemma 3.1. Now for $w \in \Gamma$ we have $w \in L_n$ and so $f^n(w) \in A_n = G_n^*$, so that $w \in I(f)$ by (7). This completes the proof of Theorem 1.1.

We do not know whether the second conclusion of Theorem 1.1 holds if f is only quasiregular on the set B_R in (1), but this seems unlikely. The difficulty is that for large n we cannot control the behaviour of f^n near $S(0, R)$ and so the component L_n in Lemma 5.3 may in principle be bounded.

6. A QUASIREGULAR MAPPING f FOR WHICH $\overline{I(f)}$ HAS A BOUNDED COMPONENT

To show that there exists a quasiregular mapping $f : \mathbb{C} \rightarrow \mathbb{C}$ such that the closure of the escaping set $I(f)$ has a bounded component, we begin by constructing a quasiconformal map g with the following properties. For each z in the punctured disc $A := \{z \in \mathbb{C} : 0 < |z| < 1\}$ the iterates g^n satisfy $\lim_{n \rightarrow \infty} |g^n(z)| = 1$, and we have $\lim_{n \rightarrow \infty} g^n(1/2) = 1$. On the other hand there exist annuli $A_n \subseteq A$ such that g maps A_n onto A_{n+1} , but with sufficient rotation that for each $z \in A_n$ infinitely many of the forward images $g^k(z)$ lie away from 1. A map h is then obtained from g by conjugation with a Möbius map L which sends 1 to ∞ , and finally h

is interpolated on a sector to ensure that the resulting function has an essential singularity at infinity.

We will use the fact that if p is quasiregular on a domain $D \subseteq \mathbb{C}$ and

$$p_z = \frac{\partial p}{\partial z} = \frac{1}{2} \left(\frac{\partial p}{\partial x} - i \frac{\partial p}{\partial y} \right)$$

is bounded below in modulus on D , and if q is continuous and such that the partial derivatives q_x, q_y are sufficiently small on D , then $p + q$ is quasiregular on D . If $0 \notin D \cup p(D)$ the same property may be applied locally to $\log p$ as a function of $\log z$.

Turning to the detailed construction, we define $a : [1, 2] \rightarrow [0, \pi/4]$ by

$$a(r) = \frac{\pi}{4} - \arcsin \left(\frac{\sqrt{2}}{2r} \right).$$

Then an application of the sine rule shows that the line segment

$$\operatorname{Re} z = 1 + \operatorname{Im} z, \quad 1 \leq |z| \leq 2,$$

is parametrized by $z = re^{ia(r)}$.

For $c > 0$ we define $g : \mathbb{C} \rightarrow \mathbb{C}$ as follows. Let $g(0) = 0$ and for $z = re^{it}$ with $r > 0$ and $-\pi \leq t \leq \pi$ set:

$$g(z) = \begin{cases} \frac{4}{3}r \exp(i(t + c|\sin t|)), & 0 < r < \frac{1}{2}; \\ \frac{1}{2-r} \exp\left(i\left(t + c|\sin t| + c(1-r)^2 \left|\sin\left(\frac{\pi}{1-r}\right)\right|\right)\right), & \frac{1}{2} \leq r < 1; \\ r \exp\left(i\left(t + c(2-r) \sin\left(\frac{|t-a(r)|}{\pi-a(r)}\pi\right)\right)\right), & 1 \leq r \leq 2, a(r) < |t|; \\ r \exp(it), & 1 \leq r \leq 2, |t| \leq a(r); \\ r \exp(it), & r > 2. \end{cases}$$

Then g is continuous on \mathbb{C} . Moreover, if c is sufficiently small then g is quasiconformal, and in particular we choose $c < \pi/4$. Note that, by the choice of $a(r)$,

$$(15) \quad g(z) = z \quad \text{if} \quad \operatorname{Re} z \geq |\operatorname{Im} z| + 1.$$

For $n \in \mathbb{N}$ we have

$$(16) \quad g\left(1 - \frac{1}{n+1}\right) = 1 - \frac{1}{n+2}.$$

For $n \in \mathbb{N}$, $n \geq 2$, we consider the annulus

$$A_n := \left\{ z \in \mathbb{C} : 1 - \frac{1}{n+1/4} < |z| < 1 - \frac{1}{n+3/4} \right\}.$$

Then $g(A_n) = A_{n+1}$.

Lemma 6.1. *For each $z \in A_n$ with $\operatorname{Re} z > 0$ there exists $k \in \mathbb{N}$ with $\operatorname{Re} g^k(z) \leq 0$.*

Proof. Let $z \in A_n$ and suppose first that $0 < t := \arg z < \pi/2$. Then

$$(17) \quad \pi > t + \frac{\pi}{2} > t + 2c \geq \arg g(z) \geq t + c \sin t \geq t + \frac{2c}{\pi}t = \left(1 + \frac{2c}{\pi}\right)t.$$

On the other hand if $-\pi/2 < t = \arg z \leq 0$ then

$$(18) \quad \frac{\pi}{2} > \arg g(z) \geq t + c|\sin t| + \frac{c\sqrt{2}}{2(m+3/4)^2} \geq \left(1 - \frac{2c}{\pi}\right)t + \frac{c'}{(m+1)^2} > -\frac{\pi}{2},$$

where $c' := \frac{1}{2}c\sqrt{2}$. In particular, (17) and (18) both hold with $\arg g(z)$ the principal argument.

Suppose then that there exists $z \in A_n$ with $\operatorname{Re} g^k(z) > 0$ for all integers $k \geq 0$, and set $t_k = \arg g^k(z) \in (-\pi/2, \pi/2)$. Then $g^k(z) \in A_{n+k}$. If there exists $k \geq 0$ with $0 < t_k < \pi/2$ then by repeated application of (17) we obtain $k' > k$ with $t_{k'} \in (\pi/2, \pi)$, a contradiction. Hence we must have $-\pi/2 < t_k \leq 0$ for all $k \geq 0$. But then repeated application of (18) gives, for large k ,

$$t_{k-1} \geq \left(1 - \frac{2c}{\pi}\right)^{k-1} t_0, \quad t_k \geq \left(1 - \frac{2c}{\pi}\right) t_{k-1} + \frac{c'}{(n+k)^2} > 0,$$

again a contradiction. \square

With the Möbius transformation

$$L(z) = \frac{1}{1-z}$$

we now consider the map $h := L \circ g \circ L^{-1}$. Then h is a quasiconformal self-map of the plane. Moreover, (15) gives $h(z) = z$ if $\operatorname{Re} L^{-1}(z) \geq |\operatorname{Im} L^{-1}(z)| + 1$, which is equivalent to $\operatorname{Re} z \leq -|\operatorname{Im} z|$, and we have

$$(19) \quad L(A_n) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \quad \text{and} \quad h(L(A_n)) = L(A_{n+1}),$$

using the fact that $g(A_n) = A_{n+1}$.

It follows from (16) that

$$(20) \quad h(n+1) = n+2 \quad \text{for} \quad n \in \mathbb{N},$$

and we deduce at once that $2 \in I(h)$. Next we show that $L(A_n) \cap I(h) = \emptyset$ for every integer $n \geq 2$. In fact, suppose that $n \geq 2$ and $u \in L(A_n) \cap I(h)$. Then there exists $j_0 \in \mathbb{N}$ such that $|h^j(u)| > 1$ for $j \geq j_0$. Put $w := h^{j_0}(u)$ and $m := n + j_0$. Then $L^{-1}(w) \in A_m$ by (19), and Lemma 6.1 gives $k \geq 0$ with $\operatorname{Re} g^k(L^{-1}(w)) \leq 0$. Since $|L(z)| \leq 1$ for $\operatorname{Re} z \leq 0$ we deduce that

$$|h^{k+j_0}(u)| = |h^k(w)| = |L(g^k(L^{-1}(w)))| \leq 1,$$

contradicting the choice of j_0 . Thus $L(A_n) \cap I(h) = \emptyset$.

Since A_2 separates $\frac{1}{2}$ from 1 it follows that 2 lies in the bounded component of the complement of $L(A_2)$, and we deduce that the component of $\overline{I(h)}$ containing 2 is bounded.

To construct a quasiregular map $f : \mathbb{C} \rightarrow \mathbb{C}$ with an essential singularity at ∞ for which the closure of $I(f)$ has a bounded component we put $f(z) = h(z)$ for $\operatorname{Re} z \geq -|\operatorname{Im} z|$ and $f(z) = z + d \exp(z^4)$ for $\operatorname{Re} z \leq -|\operatorname{Im} z| - 1$, where d is a small positive constant. In the remaining region Ω we define f by interpolation, using

$$f(z) = z - d\phi(z), \quad \phi(z) = (\operatorname{Re} z + |\operatorname{Im} z|) \exp(z^4) \quad \text{for} \quad -1 < \operatorname{Re} z + |\operatorname{Im} z| < 0.$$

Since $\exp(z^4)$ tends to 0 rapidly as z tends to infinity in Ω , it is then clear that the partial derivatives of ϕ are bounded on Ω , so that f is quasiregular on Ω because d is small.

In particular we have $f(z) = h(z)$ for $\operatorname{Re} z > 0$ and so it follows from (20) that $2 \in I(f)$, whereas $L(A_n) \cap I(f)$ is again empty using (19). Thus the component of $\overline{I(f)}$ containing 2 is bounded.

7. THE QUASIMEROMORPHIC CASE

Let f be K -quasimeromorphic in the set B_R defined in (1), with a sequence of poles tending to ∞ , and set $R_{-1} = R$. Choose x_j, D_j, R_j for $j = 0, 1, 2, \dots$ as follows. Each x_j is a pole of f , and D_j is a bounded component of the set $\{x \in B_R : R_j < |f(x)| \leq \infty\}$ which contains x_j but no other pole of f , such that D_j is mapped by f onto $\{y \in \overline{\mathbb{R}^N} : R_j < |y| \leq \infty\}$. Moreover, by choosing R_{j+1} and x_{j+1} sufficiently large, we may ensure that

$$(21) \quad |x_{j+1}| > 4R_j \quad \text{and} \quad D_{j+1} \subseteq \{x \in \mathbb{R}^N : 2R_j < |x| < \infty\} \quad \text{for } j \geq -1.$$

Since $|f(x)| = R_j$ for all $x \in \partial D_j$ we may write, for $j \geq 0$, using (21),

$$(22) \quad C_j = \{x \in D_j : f(x) \in D_{j+1}\} \subseteq \overline{C_j} \subseteq D_j.$$

Now set

$$(23) \quad X_0 = \overline{C_0}, \quad X_{j+1} = \{x \in X_j : f^{j+1}(x) \in \overline{C_{j+1}}\}.$$

Evidently X_0 is compact. Assuming that X_j is compact, it then follows that X_{j+1} is the intersection of a compact set with the closed set $f^{-j-1}(\overline{C_{j+1}})$ and so is compact. Hence the X_j form a nested sequence of compact sets. We assert that

$$(24) \quad f^j(X_j) = \overline{C_j}.$$

We clearly have $f^j(X_j) \subseteq \overline{C_j}$ by (23), and (24) is obviously true for $j = 0$, so assume the assertion for some $j \geq 0$ and take $w \in \overline{C_{j+1}}$. Since f maps D_j onto $\{y \in \overline{\mathbb{R}^N} : R_j < |y| \leq \infty\}$, it follows from (21) and (22) that there exists $v \in C_j$ with $f(v) = w$. Hence there exists $x \in X_j$ with $f^j(x) = v$ and $f^{j+1}(x) = w$, completing the induction.

Again since f maps D_j onto $\{y \in \overline{\mathbb{R}^N} : R_j < |y| \leq \infty\}$, we evidently have $C_j \neq \emptyset$ and so X_j is non-empty by (24). Hence there exists x lying in the intersection of the X_j , so that $f^j(x) \in \overline{C_j}$ and $x \in I(f)$ by (21) and (22).

8. PROOF OF LEMMA 3.2

To establish Lemma 3.2 let E and g be as in the statement and assume that $g^{-1}(E)$ is non-empty since otherwise there is nothing to prove. Note first that $g^{-1}(E)$ is a closed subset of \mathbb{R}^N by continuity. Thus

$$F = g^{-1}(E) \cup \{\infty\}$$

is a compact subset of $\overline{\mathbb{R}^N}$. In order to prove Lemma 3.2 it therefore suffices in view of Lemma 3.1 to show that F is connected. Suppose that this is not the case. Then there is a partition of F into non-empty disjoint relatively closed (and so closed) sets H_1, H_2 such that $\infty \in H_2$. Let $W = \mathbb{R}^N \setminus H_2$. Then W is an open subset of \mathbb{R}^N , and $g(W \setminus H_1) \cap E = \emptyset$. Moreover, H_1 is a closed subset of $\overline{\mathbb{R}^N}$ and so compact, and hence a compact subset of \mathbb{R}^N since $\infty \in H_2$. Thus $g(H_1)$ is compact and so a non-empty closed subset of E .

Now suppose that there exist $y_n \in E \setminus g(H_1)$ with $y_n \rightarrow \tilde{y} \notin E \setminus g(H_1)$. Since E is compact we have $\tilde{y} \in E$ and so $\tilde{y} \in g(H_1)$. Hence there exists $\tilde{x} \in H_1$ with $g(\tilde{x}) = \tilde{y}$

and for large enough n there exists x_n close to \tilde{x} with $g(x_n) = y_n \in E \setminus g(H_1)$. But then we must have $x_n \in H_1$, since $g(W \setminus H_1) \cap E = \emptyset$, and this is a contradiction. So $E \setminus g(H_1)$ is also closed, but evidently non-empty since $g(\mathbb{R}^N) \subseteq \mathbb{R}^N$ and $\infty \in E$, which contradicts the hypothesis that E is connected.

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